

# CURVATURE PROPERTIES ON SOME CLASSES OF ALMOST CONTACT MANIFOLDS WITH B-METRIC

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## Abstract

Almost contact manifolds with B-metric are considered. Of special interest are the so-called vertical classes of the almost contact B-metric manifolds. Curvature properties of these manifolds are studied. An example of 5-dimensional manifolds is constructed and characterized.

**Key words:** almost contact manifold, B-metric, indefinite metric, Lie group.

**2010 Mathematics Subject Classification:** 53C15, 53D15, 53C50.

## 1. INTRODUCTION

In this work<sup>1</sup> we continue the investigations on a manifold  $M$  with an almost contact structure  $(\varphi, \xi, \eta)$  which is equipped with a B-metric  $g$ , i.e. a pseudo-Riemannian metric of signature  $(n, n+1)$  with the opposite compatibility of the metric with the structure in comparison with the known almost contact metric structure. Moreover, the B-metric is an odd-dimensional analogue of the Norden metric on almost complex manifolds.

Recently, manifolds with neutral metrics and various tensor structures have been object of interest in theoretical physics.

The classes of the almost contact B-metric manifolds from the so-called vertical component is an object of special interest in this paper. The goal of the present work is the investigation of some problems of the geometry of such manifolds.

The paper is organized as follows. In Sec. 2 we give some necessary facts about the considered manifolds. In Sec. 3 we obtain some curvature properties on almost contact B-metric manifolds in these classes. In Sec. 4 we consider and characterize a family of 5-dimensional Lie groups as almost contact B-metric manifolds from a basic class.

## 2. ALMOST CONTACT MANIFOLDS WITH B-METRIC

Let  $(M, \varphi, \xi, \eta, g)$  be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, i.e.  $M$  is a  $(2n+1)$ -dimensional differentiable manifold with an almost contact structure  $(\varphi, \xi, \eta)$  consists of an endomorphism  $\varphi$  of the tangent bundle, a vector field  $\xi$ , its dual 1-form  $\eta$

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<sup>1</sup>Partially supported by project NI11-FMI-004 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria

as well as  $M$  is equipped with a pseudo-Riemannian metric  $g$  of signature  $(n, n+1)$ , such that the following algebraic relations are satisfied

$$(2.1) \quad \begin{aligned} \varphi\xi &= 0, & \varphi^2 &= -Id + \eta \otimes \xi, & \eta \circ \varphi &= 0, & \eta(\xi) &= 1, \\ g(\varphi x, \varphi y) &= -g(x, y) + \eta(x)\eta(y) \end{aligned}$$

for arbitrary  $x, y$  of the algebra  $\mathfrak{X}(M)$  on the smooth vector fields on  $M$ .

The associated B-metric  $\tilde{g}$  of  $g$  on  $M$  is defined by  $\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)$ . Both metrics are necessarily of signature  $(n, n+1)$ . The manifold  $(M, \varphi, \xi, \eta, \tilde{g})$  is an almost contact B-metric manifold, too.

Further,  $x, y, z, w$  will stand for arbitrary elements of  $\mathfrak{X}(M)$  unless other is specialized.

The structural group of  $(M, \varphi, \xi, \eta, g)$  is  $G \times I$ , where  $I$  is the identity on  $\text{span}(\xi)$  and  $G = \mathcal{GL}(n; \mathbb{C}) \cap \mathcal{O}(n, n)$ . More precisely,  $G$  consists of real square matrices of order  $2n$  of the following type

$$\left( \begin{array}{c|c} A & B \\ \hline -B & A \end{array} \right), \quad \begin{aligned} AA^T - BB^T &= I_n, \\ AB^T + BA^T &= O_n, \end{aligned} \quad A, B \in \mathcal{GL}(n; \mathbb{R}),$$

where  $I_n$  and  $O_n$  are the unit matrix and the zero matrix of size  $n$ , respectively.

A classification of the almost contact B-metric manifolds is given in [1]. This classification is made with respect to the  $(0,3)$ -tensor  $F$  defined by

$$(2.2) \quad F(x, y, z) = g((\nabla_x \varphi) y, z),$$

where  $\nabla$  is the Levi-Civita connection of  $g$ . The tensor  $F$  has the properties

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).$$

Let us recall the following general relations [2]:

$$(2.3) \quad F(x, \varphi y, \xi) = (\nabla_x \eta) y = g(\nabla_x \xi, y), \quad \eta(\nabla_x \xi) = 0.$$

The components of the inverse matrix of  $g$  are denoted by  $g^{ij}$  with respect to the basis  $\{e_i; \xi\}$  of the tangent space  $T_p M$  of  $M$  at an arbitrary point  $p \in M$ . The following basic 1-forms are associated with  $F$  for any  $z \in T_p M$ :

$$(2.4) \quad \theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

The basic classes in the mentioned classification are  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{11}$ . Their intersection is the class  $\mathcal{F}_0$  determined by the condition  $F = 0$ .

In [3], it is proved that the class  $\mathcal{U} = \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9$  is defined by the conditions

$$(2.5) \quad F(x, y, z) = \eta(y)F(x, z, \xi) + \eta(z)F(x, y, \xi), \quad F(\xi, y, z) = 0.$$

It is known [4], an almost contact manifold  $(M, \varphi, \xi, \eta)$  is called *normal* if the corresponding almost complex structure  $J$  on the even-dimensional manifold  $M \times \mathbb{R}$  is integrable, i.e. the Nijenhuis torsion  $[J, J](x, y) = -[x, y] + [Jx, Jy] - J[Jx, y] - J[x, Jy]$  is identically zero. According to [5], the necessary and sufficient condition  $(M, \varphi, \xi, \eta)$  to be normal is the annulment of the Nijenhuis tensor  $N$  defined by  $N(x, y) := [\varphi, \varphi](x, y) + d\eta(x, y)\xi$ , where

$[\varphi, \varphi](x, y) = \varphi^2[x, y] + [\varphi x, \varphi y] - \varphi[\varphi x, y] - \varphi[x, \varphi y]$ . It is known from [3] that the class of the normal contact B-metric manifolds is  $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ . Moreover,  $\eta$  is closed there, i.e.  $d\eta = 0$ . Therefore, also we have  $[\varphi, \varphi] = 0$  for the normal contact B-metric manifolds.

In [3], there are considered the linear projectors  $h$  and  $v$  over  $T_p M$  which split (orthogonally and invariantly with respect to the structural group) any vector  $x$  to a horizontal component  $h(x) = -\varphi^2 x$  and a vertical component  $v(x) = \eta(x)\xi$ . The decomposition  $T_p M = h(T_p M) \oplus v(T_p M)$  generates the corresponding distribution of basic tensors  $F$ , which gives the horizontal component  $\mathcal{F}_1 \oplus \mathcal{F}_2$  and the vertical component  $\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$  of the class of the normal contact B-metric manifolds.

By the additional conditions

$$(2.6) \quad F(x, y, \xi) = F(y, x, \xi) = -F(\varphi x, \varphi y, \xi),$$

it is determined the subclass  $\mathcal{U}_1 = \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \subset \mathcal{U}$ , which is the class of the normal contact manifolds from the vertical classes. This subclass is also characterized by (2.5) and  $N = 0$ . The classes  $\mathcal{F}_4$ ,  $\mathcal{F}_5$  and  $\mathcal{F}_6$  are determined in  $\mathcal{U}_1$  by the conditions  $\theta^* = 0$ ,  $\theta = 0$  and  $\theta = \theta^* = 0$ , respectively. Another direct sum of basic classes considered below is  $\mathcal{U}_2 = \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7 \subset \mathcal{U}$ , which is determined by (2.5) and

$$(2.7) \quad F(x, y, \xi) = -F(\varphi x, \varphi y, \xi).$$

By analogy with the square norm of  $\nabla J$  for an almost complex structure  $J$ , we define the *square norm of  $\nabla \varphi$*  by [6]

$$(2.8) \quad \|\nabla \varphi\|^2 = g^{ij} g^{ks} g((\nabla_{e_i} \varphi) e_k, (\nabla_{e_j} \varphi) e_s).$$

It is clear that  $\|\nabla \varphi\|^2 = 0$  is valid if  $(M, \varphi, \xi, \eta, g)$  is a  $\mathcal{F}_0$ -manifold, but the inverse implication is not always true. An almost contact B-metric manifold having a zero square norm of  $\nabla \varphi$  is called an *isotropic- $\mathcal{F}_0$ -manifold* [6]. Analogously, we consider the square norm of  $\nabla \eta$  and  $\nabla \xi$  as follows:

$$\|\nabla \eta\|^2 = g^{ij} g^{ks} (\nabla_{e_i} \eta) e_k (\nabla_{e_j} \eta) e_s, \quad \|\nabla \xi\|^2 = g^{ij} g (\nabla_{e_i} \xi, \nabla_{e_j} \xi).$$

Obviously, the relation  $\|\nabla \eta\|^2 = \|\nabla \xi\|^2$  is valid, because of (2.3).

**Proposition 2.1.** *On any almost contact B-metric manifold in  $\mathcal{U}$  the following relation holds  $\|\nabla \varphi\|^2 = -2 \|\nabla \eta\|^2 = -2 \|\nabla \xi\|^2$ .*

*Proof.* It follows immediately from (2.5). □

### 3. CURVATURE PROPERTIES ON THE CLASS $\mathcal{U}$

Let  $R = [\nabla, \nabla] - \nabla_{[, ]}$  be the curvature (1,3)-tensor of  $\nabla$ . We denote the curvature (0,4)-tensor by the same letter:  $R(x, y, z, w) = g(R(x, y)z, w)$ . It has the properties:

$$(3.1) \quad R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z), \quad \mathfrak{S}_{x,y,z} R(x, y, z, w) = 0,$$

where  $\mathfrak{S}_{x,y,z}$  denotes the cyclic sum by  $x, y, z$ .

The property  $R(x, y, \varphi z, \varphi w) = -R(x, y, z, w)$  is valid for  $R$  on a  $\mathcal{F}_0$ -manifold, because of  $\nabla \varphi = 0$  there.

The Ricci tensor  $\rho$  and the scalar curvature  $\tau$  for  $R$  as well as their associated quantities are defined respectively by

$$(3.2) \quad \rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j),$$

$$(3.3) \quad \rho^*(y, z) = g^{ij} R(e_i, y, z, \varphi e_j), \quad \tau^* = g^{ij} \rho^*(e_i, e_j).$$

Further we use the notation  $g \odot h$  for the Kulkarni-Nomizu product of two (0,2)-tensors, i.e.  $(g \odot h)(x, y, z, w) = g(x, z)h(y, w) - g(y, z)h(x, w) + g(y, w)h(x, z) - g(x, w)h(y, z)$ . Obviously,  $g \odot h$  has the properties of  $R$  in (3.1) when  $g$  and  $h$  are symmetric.

**Theorem 3.1.** *On any almost contact B-metric manifold in  $\mathcal{U}$  the following relation holds*

$$(3.4) \quad \begin{aligned} & 2R(x, y, \varphi^2 z, \varphi^2 w) + 2R(x, y, \varphi z, \varphi w) \\ & = (h \odot h)(x, y, z, w) + (h \odot h)(x, y, \varphi z, \varphi w), \end{aligned}$$

where  $h(x, y) = (\nabla_x \eta) y$ .

*Proof.* By direct computations from (2.5) and the Ricci identity  $(\nabla_x \nabla_y \varphi) z - (\nabla_y \nabla_x \varphi) z = R(x, y) \varphi z - \varphi R(x, y) z$  follows (3.4).  $\square$

**Corollary 3.2.** *On any almost contact B-metric manifold in  $\mathcal{U}_1$  we have*

$$\mathfrak{S}_{x,y,z} \{ R(x, y, \varphi^2 z, \varphi^2 w) + R(x, y, \varphi z, \varphi w) \} = 0.$$

*Proof.* When a manifold belongs to  $\mathcal{U}_1$  then the corresponding (0,2)-tensor  $h$  is symmetric and hybrid with respect to  $\varphi$ . Then the cyclic sum by  $x, y, z$  of the right-hand side of (3.4) vanishes, which accomplish the proof.  $\square$

**Corollary 3.3.** *On any almost contact B-metric manifold in  $\mathcal{U}_2$  we have*

$$\begin{aligned} & R(\varphi x, \varphi y, \varphi^2 z, \varphi^2 w) + R(\varphi x, \varphi y, \varphi z, \varphi w) \\ & - R(x, y, \varphi^2 z, \varphi^2 w) - R(x, y, \varphi z, \varphi w) = 0. \end{aligned}$$

*Proof.* Since  $g(x, \varphi y) = g(\varphi x, y)$  holds and (2.7) is valid on a manifold in  $\mathcal{U}_2$ , then using (2.3) we obtain  $\nabla_{\varphi x} \xi = \varphi \nabla_x \xi$ . The last equality implies that  $h(x, y)$  is a hybrid with respect to  $\varphi$  and then the right-hand side of (3.4) is equal to the first line of (3.3). Thus, the statement follows.  $\square$

#### 4. A LIE GROUP AS A 5-DIMENSIONAL $\mathcal{F}_6$ -MANIFOLD

Let  $G$  be a 5-dimensional real connected Lie group and let  $\mathfrak{g}$  be its Lie algebra. If  $\{e_i\}$  is a global basis of left-invariant vector fields of  $G$ , we define

an almost contact structure  $(\varphi, \xi, \eta)$  and a B-metric  $g$  on  $G$  as follows:

$$(4.1) \quad \begin{aligned} \varphi e_1 &= e_3, & \varphi e_2 &= e_4, & \varphi e_3 &= -e_1, & \varphi e_4 &= -e_2, & \varphi e_5 &= 0; \\ \xi &= e_5; & \eta(e_i) &= 0 \ (i = 1, 2, 3, 4), & \eta(e_5) &= 1; \end{aligned}$$

$$(4.2) \quad \begin{aligned} g(e_1, e_1) &= g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = g(e_5, e_5) = 1, \\ g(e_i, e_j) &= 0, \quad i \neq j, \quad i, j \in \{1, 2, 3, 4, 5\}. \end{aligned}$$

We verify that (2.1) are satisfied for  $\{e_i\}$ . Thus, we establish that the induced 5-dimensional manifold  $(G, \varphi, \xi, \eta, g)$  is an almost contact B-metric manifold.

Using the known property of the Levi-Civita connection  $\nabla$  of  $g$

$$(4.3) \quad 2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g([e_k, e_j], e_i),$$

we obtain the form of the basic tensor  $F$  as follows

$$(4.4) \quad \begin{aligned} 2F(e_i, e_j, e_k) &= g([e_i, \varphi e_j] - \varphi[e_i, e_j], e_k) + g([e_i, \varphi e_k] - \varphi[e_i, e_k], e_j) \\ &\quad + g([e_k, \varphi e_j] - [\varphi e_k, e_j], e_i). \end{aligned}$$

On any manifold in  $\mathcal{U}_2$  the properties  $F(e_i, e_j, \xi) = -F(\varphi e_i, \varphi e_j, \xi)$  and  $F(\xi, e_i, e_j) = 0$  are valid. According to (2.2) and (2.3), the last two properties are equivalent to  $\nabla_{\varphi e_i} \xi = \varphi \nabla_{e_i} \xi$  and  $\nabla_{\xi} \varphi e_i = \varphi \nabla_{\xi} e_i$ , respectively. These equalities imply the following property on a manifold in  $\mathcal{U}_2$

$$(4.5) \quad [\varphi e_i, \xi] = \varphi[e_i, \xi].$$

Using (2.6) and (4.4), the following property is valid for a manifold in  $\mathcal{U}_1$

$$(4.6) \quad \eta([e_i, e_j]) = 0.$$

To be the considered manifold in  $\mathcal{F}_6$  we set  $\theta(\xi) = \theta^*(\xi) = 0$ . Then, using (2.4) and the equality  $2F(e_i, e_j, \xi) = -g([e_i, \xi], \varphi e_j) - g([e_j, \xi], \varphi e_i)$ , which is a corollary of (4.4), we obtain conditions

$$(4.7) \quad \theta(\xi) = -g^{ij}g([e_i, \xi], \varphi e_j) = 0, \quad \theta^*(\xi) = g^{ij}g([e_i, \xi], e_j) = 0.$$

Now, let us consider the Lie algebra  $\mathfrak{g}$  on  $G$ , determined by the following non-zero commutators satisfying conditions (4.5) and (4.6):

$$\begin{aligned} [e_1, \xi] &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4, \\ [e_2, \xi] &= \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 e_4, \\ [e_3, \xi] &= -\lambda_3 e_1 - \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4, \\ [e_4, \xi] &= -\mu_3 e_1 - \mu_4 e_2 + \mu_1 e_3 + \mu_2 e_4, \end{aligned}$$

where  $\lambda_i, \mu_i \in \mathbb{R}$  ( $i = 1, 2, 3, 4$ ).

We compute  $\theta(\xi) = 2(\lambda_1 + \mu_2)$  and  $\theta^*(\xi) = 2(\lambda_3 + \mu_4)$  using (4.7) and then we determine  $\mu_2 = -\lambda_1$  and  $\mu_4 = -\lambda_3$ . Therefore, we obtain

$$\begin{aligned}
 (4.8) \quad [e_1, \xi] &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4, \\
 [e_2, \xi] &= \mu_1 e_1 - \lambda_1 e_2 + \mu_3 e_3 - \lambda_3 e_4, \\
 [e_3, \xi] &= -\lambda_3 e_1 - \lambda_4 e_2 + \lambda_1 e_3 + \lambda_2 e_4, \\
 [e_4, \xi] &= -\mu_3 e_1 + \lambda_3 e_2 + \mu_1 e_3 - \lambda_1 e_4.
 \end{aligned}$$

It is easy to obtain that the other commutators  $[e_i, e_j]$  ( $i, j \in \{1, 2, 3, 4\}$ ) are zero.

We verify immediately that the Jacobi identity of  $[e_i, e_j]$  for all  $i, j \in \{1, 2, 3, 4, 5\}$  is satisfied.

Using (4.4) and (4.8), we obtain that the non-zero components of  $F$  are:

$$\begin{aligned}
 (4.9) \quad F_{115} &= -F_{225} = -F_{335} = F_{445} = \lambda_3, \quad F_{135} = -F_{245} = \lambda_1, \\
 F_{145} &= F_{235} = \frac{1}{2}(\lambda_2 + \mu_1), \quad F_{125} = -F_{345} = \frac{1}{2}(\lambda_4 + \mu_3),
 \end{aligned}$$

where  $F_{ij5} = F(e_i, e_j, \xi)$ ,  $i, j \in \{1, 2, 3, 4\}$ .

**Theorem 4.1.** *Let  $(G, \varphi, \xi, \eta, g)$  be the almost contact B-metric manifold, determined by (4.1), (4.2) and (4.8). Then it belongs to the class  $\mathcal{F}_6$ .*

*Proof.* We check directly that the components in (4.9) satisfy conditions (2.5), (2.6) and  $\theta = \theta^* = 0$  for  $\mathcal{F}_6$ . Therefore we establish that the corresponding manifold  $(G, \varphi, \xi, \eta, g)$  belongs to  $\mathcal{F}_6$ . It is a  $\mathcal{F}_0$ -manifold if and only if the condition  $\lambda_1 = \lambda_2 + \mu_1 = \lambda_3 = \lambda_4 + \mu_3 = 0$  holds.  $\square$

Using (4.3) and (4.8), we obtain the components of  $\nabla$ :

$$\begin{aligned}
 (4.10) \quad \nabla_{e_1} e_1 &= -\nabla_{e_2} e_2 = -\nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\lambda_1 \xi, \quad \nabla_\xi \xi = 0, \\
 \nabla_{e_1} e_2 &= \nabla_{e_2} e_1 = -\nabla_{e_3} e_4 = -\nabla_{e_4} e_3 = -\frac{1}{2}(\lambda_2 + \mu_1), \\
 \nabla_{e_1} e_3 &= -\nabla_{e_2} e_4 = \nabla_{e_3} e_1 = -\nabla_{e_4} e_2 = \lambda_3 \xi, \\
 \nabla_{e_1} e_4 &= \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \nabla_{e_4} e_1 = \frac{1}{2}(\lambda_4 + \mu_3), \\
 \nabla_{e_1} \xi &= \lambda_1 e_1 + \frac{1}{2}(\lambda_2 + \mu_1) e_2 + \lambda_3 e_3 + \frac{1}{2}(\lambda_4 + \mu_3) e_4, \\
 \nabla_{e_2} \xi &= \frac{1}{2}(\lambda_2 + \mu_1) e_1 - \lambda_1 e_2 + \frac{1}{2}(\lambda_4 + \mu_3) e_3 - \lambda_3 e_4, \\
 \nabla_{e_3} \xi &= -\lambda_3 e_1 - \frac{1}{2}(\lambda_4 + \mu_3) e_2 + \lambda_1 e_3 + \frac{1}{2}(\lambda_2 + \mu_1) e_4, \\
 \nabla_{e_4} \xi &= -\frac{1}{2}(\lambda_4 + \mu_3) e_1 + \lambda_3 e_2 + \frac{1}{2}(\lambda_2 + \mu_1) e_3 - \lambda_1 e_4, \\
 \nabla_\xi e_1 &= -\frac{1}{2}(\lambda_2 - \mu_1) e_2 - \frac{1}{2}(\lambda_4 - \mu_3) e_4, \\
 \nabla_\xi e_2 &= \frac{1}{2}(\lambda_2 - \mu_1) e_1 + \frac{1}{2}(\lambda_4 - \mu_3) e_3, \\
 \nabla_\xi e_3 &= \frac{1}{2}(\lambda_4 - \mu_3) e_2 - \frac{1}{2}(\lambda_2 - \mu_1) e_4, \\
 \nabla_\xi e_4 &= -\frac{1}{2}(\lambda_4 - \mu_3) e_1 + \frac{1}{2}(\lambda_2 - \mu_1) e_3.
 \end{aligned}$$

Let us denote  $R_{ijkl} = R(e_i, e_j, e_k, e_l)$  and  $\rho_{jk} = \rho(e_j, e_k)$ . According to (4.2) and (4.10), we obtain

$$\begin{aligned}
 R_{5115} &= -R_{5335} = \rho_{11} = -\rho_{33} \\
 &= -\lambda_1^2 + \lambda_3^2 - \frac{3}{4}(\lambda_2^2 - \lambda_4^2) + \frac{1}{4}(\mu_1^2 - \mu_3^2) - \frac{1}{2}(\lambda_2\mu_1 - \lambda_4\mu_3), \\
 R_{5225} &= -R_{5445} = \rho_{22} = -\rho_{44} \\
 &= -\lambda_1^2 + \lambda_3^2 + \frac{1}{4}(\lambda_2^2 - \lambda_4^2) - \frac{3}{4}(\mu_1^2 - \mu_3^2) - \frac{1}{2}(\lambda_2\mu_1 - \lambda_4\mu_3), \\
 (4.11) \quad R_{5125} &= -R_{5345} = \rho_{12} = -\rho_{34} = \lambda_1(\lambda_2 - \mu_1) - \lambda_3(\lambda_4 - \mu_3), \\
 R_{5145} &= R_{5235} = \rho_{14} = \rho_{23} = -\lambda_1(\lambda_4 - \mu_3) - \lambda_3(\lambda_2 - \mu_1), \\
 R_{5135} &= \rho_{13} = 2\lambda_1\lambda_3 + \frac{3}{2}\lambda_2\lambda_4 - \frac{1}{2}\mu_1\mu_3 + \frac{1}{2}(\lambda_2\mu_3 + \lambda_4\mu_1), \\
 R_{5245} &= \rho_{24} = 2\lambda_1\lambda_3 - \frac{1}{2}\lambda_2\lambda_4 + \frac{3}{2}\mu_1\mu_3 + \frac{1}{2}(\lambda_2\mu_3 + \lambda_4\mu_1), \\
 2\rho_{55} &= -8(\lambda_1^2 - \lambda_3^2) - 2(\lambda_2 + \mu_1)^2 + 2(\lambda_4 + \mu_3)^2 = \tau.
 \end{aligned}$$

The rest of the non-zero components of  $R$  and  $\rho$  are determined by (4.11) and the properties  $R_{ijkl} = R_{klij}$ ,  $R_{ijkl} = -R_{jikl} = -R_{ijlk}$  and  $\rho_{jk} = \rho_{kj}$ . Moreover, the associated scalar curvature is

$$(4.12) \quad \tau^* = 8\lambda_1\lambda_3 + 2(\lambda_2 + \mu_1)(\lambda_4 + \mu_3).$$

It is easy to check that the components of  $R$  in (4.11) satisfy the equalities in Theorem 3.1, Corollary 3.2 and Corollary 3.3.

**Theorem 4.2.** *Let  $(G, \varphi, \xi, \eta, g)$  is the  $\mathcal{F}_6$ -manifold determined by (4.1), (4.2) and (4.8). Then the following properties are equivalent:*

- (i)  $(G, \varphi, \xi, \eta, g)$  is an isotropic- $\mathcal{F}_0$ -manifold;
- (ii)  $(G, \varphi, \xi, \eta, g)$  is scalar flat;
- (iii)  $4(\lambda_3^2 - \lambda_1^2) - (\lambda_2 + \mu_1)^2 + (\lambda_4 + \mu_3)^2 = 0$ .

*Proof.* By virtue of (2.8) and (4.9) we get  $\|\nabla\varphi\|^2 = 4(\lambda_3^2 - \lambda_1^2) - (\lambda_2 + \mu_1)^2 + (\lambda_4 + \mu_3)^2$ . Having in mind Proposition 2.1, the latter equality and the last line of (4.11), we have

$$\begin{aligned}
 \tau &= 2\|\nabla\varphi\|^2 = -4\|\nabla\eta\|^2 = -4\|\nabla\xi\|^2 \\
 &= 8(\lambda_3^2 - \lambda_1^2) - 2(\lambda_2 + \mu_1)^2 + 2(\lambda_4 + \mu_3)^2.
 \end{aligned}$$

Then the statement follows immediately.  $\square$

Let us remark, there exists such a  $\mathcal{F}_6$ -manifold which is a scalar flat isotropic- $\mathcal{F}_0$ -manifold but it is not a  $\mathcal{F}_0$ -manifold. For example, such a case is when  $\lambda_1 = \lambda_3 \neq 0$ ,  $\mu_1 = -\lambda_2$  and  $\mu_3 = -\lambda_4$ .

According to (4.2), (3.2), (4.11) and (4.12), we obtain

**Proposition 4.3.** *The  $\mathcal{F}_6$ -manifold  $(G, \varphi, \xi, \eta, g)$ , determined by (4.1), (4.2) and (4.8), is almost Einsteinian, i.e.  $\rho_{ij} = \nu g_{ij} + \tilde{\nu} \tilde{g}_{ij}$ , if and only if the following conditions are valid*

$$\mu_1 = \lambda_2, \quad \mu_3 = \lambda_4, \quad 3(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2(\lambda_1\lambda_3 + \lambda_2\lambda_4) = 0.$$

*In this case we have*

$$\begin{aligned} \tau &= 8\nu = -8(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2), & \tau^* &= -4\tilde{\nu} = 8(\lambda_1\lambda_3 + \lambda_2\lambda_4), \\ \text{i.e. } 3\tau + 2\tau^* &= 0 \text{ and } \rho_{ij} = \frac{\tau}{8}(g_{ij} + 3\tilde{g}_{ij}). & & \square \end{aligned}$$

Bearing in mind Theorem 4.2 and Proposition 4.3, we have immediately

**Corollary 4.4.** *The  $\mathcal{F}_6$ -manifold  $(G, \varphi, \xi, \eta, g)$ , determined by (4.1), (4.2) and (4.8), is a scalar flat almost Einsteinian isotropic- $\mathcal{F}_0$ -manifold, if and only if the following condition is valid  $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = 0$ .  $\square$*

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